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Intrinsic Dirac Operators in $\mathbb{C}^n$

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INTRODUCTION

In their book, [6], Gürlebeck and Sprössig use Clifford algebras and

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other closely related problems. The use of Clifford algebras and Dirac operators essentially enables these authors to show that techniques involving complex analysis used in the real, two-dimensional setting to solve such problems readily extend to higher dimensions. As pointed out in [6], anyone familiar with Clifford algebra techniques can see that the methods described in [6] carry over to R^n, for each integer n greater than one.

The methods employed in [6] makes essential use of the Plemelj formulae and singular Cauchy transforms arising in Clifford analysis [7]. In more recent works involving Clifford analysis [4, 5, 8, 9], the dependence on Liapunov surfaces is dropped in favor of more general surfaces and domains; for instance, Lipschitz domains.

The main purpose of this paper is to show how analysis of the Dirichlet problem and Poisson's equation can be posed and soved within $\mathbb{C}^n$, again using Clifford algebras. In previous work (see, for instance, [10–13] and references therein), we have shown that many aspects of Clifford analysis extend to $\mathbb{C}^n$, and link up with the study of several complex variables. One assumption underlying all of that work is that all functions considered are holomorphic. However, many problems considered in real Clifford analysis, over domains in R^n, do not depend on that assumption. This is particularly true of much of the analysis developed in [6].

Our previous work in the several complex variable setting has considerably relied upon the use of certain types of real n-dimensional manifolds lying in $\mathbb{C}^n$. These manifolds are space-like. Intuitively, this means that they carry over most of the properties possessed by domains in R^n that are needed to develop real Clifford analysis or classical potential theory. Associated to each such manifold is a domain in $\mathbb{C}^n$ called a cell of

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harmonicity. Any holomorphic function defined in a neighborhood of the underlying manifold, and satisfying the Dirac equation in $\mathbb{C}^n$ automatically has a holomorphic extension to the associated cell of harmonicity, or a covering of that cell. When the underlying manifold is a domain in R^n, then any function defined on the domain and satisfying the Dirac operator in R^n extends to the associated cell of harmonicity, or a covering space of the cell.

This last observation suggests that it would be desirable to be able to distinguish the functions defined on the special types of real manifolds mentioned here, which may be extended to solutions of the Dirac equation over the associated cells of harmonicity. To do this, we introduce a Dirac operator which is intrinsic to the underlying manifold. This idea is due to Coifman. We show that functions annihilated by this operator extend to solutions to the Dirac equation in $\mathbb{C}^n$. Having introduced intrinsic Dirac operators, we are no longer tied to a dependence on holomorphy, and so solving Dirichlet problems and Poisson's equation, together with other related problems, may be attacked in this more general setting. This approach also enables us to introduce an analogue of harmonic measure over the boundaries of the manifolds considered here. This analysis involves a detailed study of the Hilbert module of L^2-integrable functions over M which extend to solutions to the Dirac equation in $\mathbb{C}^n$. Here M is one of the special manifolds that we mentioned earlier. However, for general M the quadratic form that we need to use is related to, but not identical to, the usual Hilbert space inner product.

Aspects of Real Clifford Analysis

We begin with some basic aspects of Cliffor algebras. First consider the real, n-dimensional vector space R^n, with orthonormal basis $\{e_j\}_{j=1}^n$. It is well-known that from R^n one can construct a real, 2^n-dimensional algebra A_n with basis elements $1, e_1, ..., e_n, ..., e_{j_1} ..., e_{j_r}, ..., e_1 \cdots e_n$, where $j_1 < \cdots < j_r$, and $1 \leqslant r \leqslant n$. Moreover, the elements $e_1, ..., e_n$ satisfy the anticommunication relation $e_j e_k + e_k e_j = -2\delta_{jk}$, where δ_{jk} is the Kroneker delta function, and $1 \leqslant j, k \leqslant n$. The algebra A_n is an example of a Clifford algebra. One important property of this algebra is that each vector $\underline{x} \in R^n \backslash \{0\} \subseteq A_n$ has a multiplicative inverse within A_n. This inverse is the vector $-\underline{x}/\|\underline{x}\|^2$.

The norm of an element $X = x_0 + \cdots + x_{1, ..., n} e_1 \cdots e_n \in A_n$ is defined to be $\|X\| = (x_0^2 + \cdots + x_{1, ..., n}^2)^{1/2}$. As A_n is a finite-dimensional algebra, then there is a contant $C_n \in R^+$ such that

$$\|XY\| \leqslant C_n \|X\| \, \|Y\|,$$

for each $X, Y \in A_n$.

We now introduce the homogeneous Dirac operator in R^n. This is the operator

$$D=\sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}.$$

DEFINITION 1. Suppose that U is a domain in R^n and $f\colon U \to A_n$ is a C^1-function, then f is called a left-regular, or left-monogenic function, if $Df(\underline{x})=0$ for each $\underline{x} \in U$.

The function f is said to be right-regular, or right-monogenic, if $f(\underline{x})\,D=0$ for each $\underline{x} \in U$. Here, $f(\underline{x})\,D$ means $\sum_{j=1}^{n} (\partial f/\partial x_j)(\underline{x})\, e_j$.

Properties of functions which are left-regular or right-regular have been studied by many authors (see, for instance, [2, 5, 6, 9]) and the study of these functions is called Clifford analysis.

It may be observed that as $D^2=-\sum_{j=1}^{n} (\partial^2/\partial x_j^2)$, then each left- or right-regular function is harmonic. The function $G(\underline{x}-\underline{y})=(\underline{x}-\underline{y})/\|\underline{x}-\underline{y}\|^n$ is an example of a function which is both left- and right-regular. In fact, the function $G(\underline{x}-\underline{y})$ is a generalization of the Cauchy kernel from one variable complex analysis.

The following generalization of Cauchy's integral formula was first established for the case $n=3$ in [3].

THEOREM 1. *Suppose $f\colon U \to A_n$ is a left-regular function, and M is a compact, n-dimensional C^1-manifold lying in U. Then for each point $\underline{y} \in \mathring{M}$ we have*

$$f(\underline{y})=\frac{1}{\omega_n}\int_{\partial M} G(\underline{x}-\underline{y})\,\underline{n}(\underline{x})\,f(\underline{x})\,d(\partial M), \tag{1}$$

where ω_n is the surface area of the unit sphere in R^n, $\underline{n}(\underline{x})$ is the outward-pointing normal vector at $\underline{x} \in \partial M$, and $d(\partial M)$ is the Lebesgue measure on ∂M.

It is clear that Theorem 1 also holds if we assume that ∂M is piecewise C^1. In fact, [14, p. 368], a theorem due to Rademacher, says that given a domain $U' \subseteq R^{n-1}$ for each $m \in N$, a Lipschitz continuous function $g\colon U' \to R^m$ is differentiable almost everywhere. Consequently, a C^0 surface Q lying in R^n has a tangent space at almost every point, provided Q has an atlas consisting of Lipschitz continuous charts. If Q is such a surface, then Q is called a Lipschitz surface.

DEFINITION 2. A domian U lying in R^n is called a Lipschitz domain if the set $\overline{U}\backslash U$ is a Lipschitz surface.

As almost every point on a Lipschitz surface has a tangent space, then the integral formula (1) remains valid if we assume that ∂M is a compact Lipschitz surface.

It is straightforward to construct left-regular functions. In fact, from Hölder's inequality and other standard arguments we have:

THEOREM 2. *Suppose that Q is an oriented Lipschitz surface in R^n, and $g: Q \to A_n$ is an L^p-integrable function with respect to Lebesgue measure on Q, where $1 \leqslant p < \infty$, then on each component of $R^n \backslash Q$ the function*

$$\frac{1}{\omega_n} \int_Q G(\underline{x} - \underline{y})\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ \tag{2}$$

is a left-regular function, where dQ stands for the Lebesgue measure on Q.

The special case where $g(x)$ is real-valued, and $Q = R^{n-1}$, has been investigated in detail in [15].

We shall also be interested in the integral (2) when $y \in Q$. In this case, the integral is interpreted as a principal value integral, whenever it is defined.

By the same proof as that given for Proposition 1 in [13] we have:

PROPOSITION 1. *Suppose that Q is an oriented Lipschitz surface in R^n, and $g: Q \to A_n$ is a Hölder continuous function with compact support, then for each $\underline{y}$ with a tangent space on Q we have that the integral*

$$P.V. \ \frac{1}{\omega_n} \int_Q G(\underline{x} - \underline{y})\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ \tag{3}$$

is bounded.

One important feature of Proposition 1 and [13, Proposition 1] is that the assumption made by some authors [6, 7] that Q be a Liapunov surface has been abandoned.

By similar arguments to those used in [7], Proposition 1 may now be used to show that if g has Hölder exponent $\alpha \in (0, 1)$, then the integral (3) defines a Hölder continuous function of exponent α on Q. We denote this function by $T_Q(g)(\underline{y})$. If the function g is Lipschitz continuous, then we may also mimic [7] to show that $T_Q(g)(\underline{y})$ is Hölder continuous for each $\alpha \in (0, 1)$.

We may also follow [7] and Proposition 1 to derive the following analogues of the classical Plemelj formulae:

THEOREM 3. *Suppose that Q is an oriented Lipschitz surface, and $g: Q \to A_n$ is a Hölder continuous function with compact support and with* *exponent* $\alpha \in (0, 1]$. *Suppose also that* Q' *is the dense subset of* Q *consisting of the points with tangent spaces on* Q. *Then*:

(a) *If* $\lambda_{\underline{y}}: R^n \backslash Q$ *is a* C^1*-function with* $\lim_{t \to 0} \lambda_{\underline{y}}(t) = \underline{y} \in Q'$, *and* $\langle \lambda_{\underline{y}}(t), \underline{n}(\underline{y}) \rangle > 0$ *for each* $t > 0$, *then*

$$\lim_{t \to 0} \frac{1}{\omega_n} \int_Q G(\underline{x} - \lambda_{\underline{y}}(t))\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ = \frac{1}{2} g(\underline{y}) + T_Q(g)(\underline{y}).$$

(b) *If* $\mu_{\underline{y}}: (0, 1] \to R^n \backslash Q$ *is a* C^1*-function with* $\lim_{t \to 0} \mu_{\underline{y}}(\underline{t}) = \underline{y} \in Q'$, *and* $\langle \mu_{\underline{y}}(t), n(\underline{y}) \rangle < 0$ *for each* $t > 0$, *then*

$$\lim_{t \to 0} \frac{1}{\omega_n} \int_Q G(\underline{x} - \mu_{\underline{y}}(t))\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ = -\frac{1}{2} g(\underline{y}) + T_Q(g)(\underline{y}).$$

Following [7], it is now straightforward to deduce that $T_Q(T_Q(g))(\underline{y}) = g(\underline{y})$ for all $\underline{y} \in Q$.

Although we have chosen to describe these last few results using Hölder continuous functions, it should be pointed out that Proposition 1 holds in a much wider context; see [5, 9]. However, the setting so far described will suffice for our need in this paper.

We now turn to look at some special types of inhomogeneous Dirac operators. First, suppose that $B: U \to R$ is a C^1-function, and let b denote the gradient function $\sum_{j=1}^n e_j(\partial B/\partial x_j)$.

Definition 3. A C^1-function $h: U \to A_n$ is said to be left-regular with respect to the potential B if $Dh(\underline{x}) - b(\underline{x})\, h(\underline{x}) = 0$ for all $\underline{x} \in U$. The function $h(\underline{x})$ is right-regular with respect to B if $h(\underline{x})\, D - h(\underline{x})\, b(\underline{x}) = 0$.

The function $b(\underline{x})$ appearing in the previous definition is called a gradient potential.

We shall denote the right A_n-module of left-regular functions defined on the domain U by $\Gamma_1(U, A_n)$, while we denote the right A_n-module of left-regular functions with respect to the potential B defined over U by $\Gamma_{1,B}(U, A_n)$. We may similarly introduce the left A_n-modules $\Gamma_r(U, A_n)$ and $\Gamma_{r,B}(U, A_n)$ of, respectively, right-regular functions and right-regular functions with respect to B.

We have the following simple isomorphism:

Proposition 2. *The modules* $\Gamma_1(U, A_n)$ *and* $\Gamma_{1,B}(U, A_n)$ *are canonically isomorphic.*

Proof. If $f(\underline{x}) \in \Gamma_1(U, A_n)$, then clearly $f(\underline{x})\, e^{-B(\underline{x})} \in \Gamma_{1,B}(U, A_n)$. Conversely, if $h(\underline{x}) \in \Gamma_{1,B}(U, A_n)$, then $h(\underline{x})\, e^{B(\underline{x})} \in \Gamma_1(U, A_n)$. ∎

Although the isomorphism described in the previous proposition is extremely simple, it is worth bearing in mind that the product rule for differentiation does not allow so simple an isomorphism between spaces of solutions to Laplace's equation and the Poisson equation $\sum_{j=1}^{n} (\partial^2 h/\partial x_j^2)(\underline{x}) = c(\underline{x})\, h(\underline{x})$, even for every simple choices of $c(\underline{x})$. This isomorphism, described in proposition 2, does not really rely on Clifford algebras. Suppose that A is a Banach algebra and that V is a finite-dimensional, real-vector subspace of A. Suppose that V is spanned by the vectors $k_1, ..., k_p$, then we can introduce the differential operator $\sum_{j=1}^{p} k_j(\partial/\partial x_j) = D_v$.

Definition 4. Suppose that U' is a domain in V, then a C^1-function $f'\colon U' \to A$ is called left-regular with respect to D_v if $D_v f' = 0$. If $f' D_v = \sum_{j=1}^{p} (\partial f'/\partial x_j)\, k_j = 0$, then f' is said to be right-regular with respect to D_v.

The right A-module of left-regular functions over U' with respect to D_v is denoted by $\Gamma_1(U', A)$, while $\Gamma_r(U', A)$ denotes the left A-module of right-regular functions over U' with respect to D_v.

Definition 5. Given a C^1-function $C\colon U' \to R$, a C^1-function $h'\colon U' \to A$ is called left-regular with respect to D_v and C if

$$D_v h'(\underline{x}) - \sum_{j=1}^{p} k_j \frac{\partial C}{\partial x_j}(\underline{x})\, h'(\underline{x}) = 0 \tag{4}$$

for each $\underline{x} \in U'$.

Equation (4) is a natural generalization of some simpler Vekua systems of equations in the complex plane.

The right A-module of functions on U' which are left-regular with respect to $k_1, ..., k_p$ and C is denoted by $\Gamma_{1,C}(U', A)$. By identical reasoning to that which was used to establish proposition 2 we have:

Proposition 3. *The modules $\Gamma_1(U', A)$ and $\Gamma_{1,C}(U', A)$ are canonically isomorphic.*

A special example of equation (4) arises when A is the real algebra spanned by 1, ie_1, ie_2, ie_3, e_1e_2, e_2e_3, $ie_1e_2e_3$, where the elements e_1, e_2 and e_3 generate the algebra A_3. Moreover, V is the space spanned by 1, ie_1, ie_2 and ie_3, and $C(\underline{x}) = -mx_0$ with $m \in R^+$. In this case, equation (4) becomes the Dirac equation with mass,

$$\left(\frac{\partial}{\partial x_0} + i \sum_{j=1}^{3} e_j \frac{\partial}{\partial x_j}\right) h'(\underline{x}) = mh'(\underline{x}),$$

arising in mathematical physics.

From Proposition 3 and Stokes' Theorem we have the following version of Cauchy's Theorem:

THEOREM 4. *Suppose that $h': U' \to A$ is left-regular with respect to D_v and C: $U' \to R$, and l': $U' \to A$ is right-regular with respect to D_v and $-C$. Suppose also that $\{k_j\}_{j=1}^{p}$ is an orthonormal basis for V. Then if M is a compact, p-dimensional manifold lying in U', we have that*

$$\int_{\partial M} h'(\underline{x})\, \underline{n}(\underline{x})\, l'(\underline{x})\, d(\partial M) = 0,$$

where $n(\underline{x})$ is the outward-point unit vector at $\underline{x} \in \partial M$.

Returning to the case where $A = A_n$, we have from Theorems 1 and 4 the following generalized Cauchy integral formula:

THEOREM 5. *Suppose that h: $U \to A_n$ is left-regular with respect to $B(\underline{x})$. Then for each compact, n-dimensional manifold M lying in U and each $\underline{x}_0 \in \mathring{M}$ then*

$$h(\underline{x}_0) = \frac{1}{\omega_n} \int_{\partial M} W_B(\underline{x}, \underline{x}_0)\, \underline{n}(\underline{x})\, g(\underline{x})\, d(\partial M),$$

where $W_B(\underline{x}, \underline{x}_0) = G(\underline{x} - \underline{x}_0)\, e^{B(\underline{x}_0) - B(\underline{x})}$.

Using Theorem 5 we can easily extend the arguments described in [6, 7] to deduce:

THEOREM 6. *Suppose that h: $U \to A_n$ is a bounded, C^1-function and U is a bounded domain. Then*

$$H: U \to A_n: H(\underline{x}_0) = \frac{1}{\omega_n} \int_U W_B(\underline{x}, \underline{x}_0)\, h(\underline{x})\, dx^n$$

satisfies the equation

$$\sum_{j=1}^{n} DH(\underline{x}_0) - b(\underline{x}_0)\, H(\underline{x}_0) = -h(\underline{x}_0).$$

Theorems 5 and 6 tell us that many basic results which hold for the homogeneous differential operator D also hold for the inhomogeneous differential operator $D - b(\underline{x})$. In particular, using the notation employed in Theorem 3, we have:

THEOREM 7. *Suppose that Q is an oriented Lipschitz surface in R^n, and $g\colon Q \to A_n$ is such that $g(\underline{x})\, e^{B(\underline{x})}$ is a Hölder continuous function with exponent $\alpha \in (0, 1)$, and has compact support. Then*:

(a) *The integral* $P.V.\,(1/\omega_n) \int_Q W_B(\underline{x}, \underline{y})\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ$ *is bounded on the set Q' and the function* $e^{B(\underline{y})} P.V.\,(1/\omega_n) \int_Q W_B(\underline{x}, \underline{y})\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ$ *is Hölder continuous with exponent α.*

$$\text{(b)}\quad \lim_{t \to 0} \frac{1}{\omega_n} \int_Q W_B(\underline{x}, \lambda_{\underline{y}}(t))\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ$$

$$= \frac{1}{2} g(\underline{y}) + P.V. \frac{1}{\omega_n} \int_Q W_B(\underline{x}, \underline{y})\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ, \quad \textit{for each } \underline{y} \in Q'.$$

$$\text{(c)}\quad \lim_{t \to 0} \frac{1}{\omega_n} \int_Q W_B(\underline{x}, \mu_{\underline{y}}(t))\, \underline{n}(\underline{x})\, g(\underline{x})\, dQ$$

$$= -\frac{1}{2} g(\underline{y}) + P.V. \frac{1}{\omega_n} \int_Q W_B(\underline{x}, \underline{y})\, n(\underline{x})\, g(\underline{x})\, dQ,$$

for each $\underline{y} \in Q'$. ▮

ASPECTS OF COMPLEX CLIFFORD ANALYSIS

Instead of considering real Clifford algebras, we now turn to look at complex Clifford algebras. The complex Clifford algebra $A_n(\mathbb{C})$ is simply the complex extension of A_n. The norm of an element $Z = z_0 + \cdots + z_{1\ldots n} e_1 \cdots e_n \in A_n(\mathbb{C})$ is defined to be $\|Z\| = (|z_0|^2 + \cdots + |z_{1\ldots n}|^2)^{1/2}$. It is easily seen that $(A_n(\mathbb{C}), \|\cdot\|)$ is a Banach algebra.

The complex subspace of $A_n(\mathbb{C})$ spanned by $e_1, \ldots, e_n$ will be denoted by $\mathbb{C}^n$ (so, we may consider the vector space $\mathbb{C}^n$ as being embedded in $A_n(\mathbb{C})$). Although every non-zero vector in $R^n \subseteq A_n$ is invertible, it is not the case that every non-zero vector $\underline{z} = z_1 e_1 + \cdots + z_n e_n \in \mathbb{C}^n$ has a multiplicative inverse in $A_n(\mathbb{C})$; for instance, $e_1 + ie_2$ is not invertible.

DEFINITION 6. The set $N(\underline{0}) = \{\underline{z} \in \mathbb{C}^n : \underline{z}^2 = 0\}$ is called the null cone at $\underline{0}$. For $\underline{z}_1 \in \mathbb{C}^n$ the set $N(\underline{z}_1) = \{\underline{z} \in \mathbb{C}^n : (\underline{z} - \underline{z}_1)^2 = 0\}$ is called the null cone at $\underline{z}_1$.

We shall be interested in special types of holomorphic functions defined within $\mathbb{C}^n$.

DEFINITION 7 [11]. Suppose that Ω is a domain in $\mathbb{C}^n$ and $f: \Omega \to A_n(\mathbb{C})$ is a holomorphic function which satisfies the differential equation

$$\sum_{j=1}^{n} e_j \frac{\partial f}{\partial z_j}(\underline{z}) = 0,$$

then $f(\underline{z})$ is called a complex left-regular function.

A similar definition can be given for complex right-regular functions. When n is even, then $G^*(\underline{z}) = \underline{z}(\underline{z}^2)^{-n/2}$, for $\underline{z} \in \mathbb{C}^n \backslash N(\underline{0})$, is an example of a function which is both complex left- and complex right-regular.

It is straightforward to see that Definition 7 may be extended so that Ω is no longer a domain lying in $\mathbb{C}^n$, but is a Riemann surface covering a domain Ω' in $\mathbb{C}^n$. Consequently, it may be determined, [13], that when n is odd, the function $\underline{x}/\|\underline{x}\|^n$ extends to a function which is both complex left- and right-regular, and is defined on a Riemann surface which double covers $\mathbb{C}^n \backslash N(\underline{0})$. We again denote this function by $G_n^+(\underline{z})$. We shal denote the complex Dirac operator $\sum_{j=1}^n e_j(\partial/\partial z_j)$ by $D_{\mathbb{C}}$.

In [11] , we introduce particular types of real, n-dimensional manifolds lying in $\mathbb{C}^n$, and we show that many aspects of real Clifford analysis carry through to $\mathbb{C}^n$ via these manifolds. We now reintroduce these manifolds:

DEFINITION 8. A compact, real, n-dimensional, C^1-manifold M lying in $\mathbb{C}^n$ is called a manifold of type one if for each $\underline{z} \in M$ we have:

(i) $N(\underline{z}) \cap M = \{\underline{z}\}$,

(ii) $N(\underline{z}) \cap TM_{\underline{z}} = \{\underline{z}\}$,

where $TM_{\underline{z}}$ is the tangent space to M at $\underline{z}$.

The assumption in the previous definition that M is a C^1-manifold may be weakened to assume that $\mathring{M}$ is C^1, and that ∂M is a Lipschitz manifold. We again call such manifolds manifolds of type one. The abundance of such manifolds follows from arguments given in [12]. In particular, if U is a bounded Lipschitz domain lying in R^n, then $\bar{U} \subseteq \mathbb{C}^n$ is a manifold of type one.

Associated with each manifold of type one is a special type of domain in $\mathbb{C}^n$ called a cell of harmonicity:

DEFINITION 9 [11]. Suppose $M \subseteq \mathbb{C}^n$ is a manifold of type one. Then the component of $\mathbb{C}^n \backslash \bigcup_{\underline{z} \in \partial M} N(\underline{z})$ containing $\mathring{M}$ is called a cell of harmonicity. This cell of harmonicity is denoted by M^+.

By considering the continuous map $A: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$: $A(\underline{z}_1, \underline{z}_2) = (\underline{z}_1 - \underline{z}_2)^2$, then it follows from the compactness of ∂M that M^+ is a domain in $\mathbb{C}^n$.

When M is a subset of R^n, then the associated cell of harmonicity corresponds to one of the bounded cells of harmonicity discussed in [1].

We now introduce the differential form

$$W\underline{z} = \sum_{j=1}^{n} (-1)^j e_j \, d\hat{z}_j,$$

where $d\hat{z}_j = dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n$, belongs to the complex extension $\Lambda(\mathbb{C}^n)$ of the real alternating algebra $\Lambda(R^n)$. From [11] we have:

THEOREM 8. *Suppose that n is even and greater than two, and $M \subseteq \mathbb{C}^n$ is a manifold of type one lying in a domain Ω. Then if $f: \Omega \to A_n(\mathbb{C})$ is a complex left-regular function, we have for each $\underline{z}_0 \in \mathring{M}$*

$$f(\underline{z}_0) = \frac{1}{\omega_n} \int_{\partial M} G^+(\underline{z} - \underline{z}_0) \, W\underline{z} \, f(\underline{z}),$$

and $f(\underline{z})$ has a unique continuation to a complex left-regular function on $\Omega \cup M^+$.

Now when $W\underline{z}$ is restricted to R^n, we obtain the differential form $W\underline{x} = \sum_{j=1}^{n} (-1)^j e_j \, d\hat{x}_j$, and when $\partial M \subseteq R^n$, this form is equivalent to vector-valued measure $\underline{n}(\underline{x}) \, d(\partial M)$ arising in expression (1).

Suppose now that N is a real, oriented $(n-1)$-dimensional C^1-manifold lying in $\mathbb{C}^n$. Then by taking any local C^1-parameterization of N, we may deduce, [13], that there is a C^1-map

$$\underline{n}: N \to \mathbb{C}^n \tag{5}$$

such that the form $W\underline{z}$ restricted to N is equivalent to $\underline{n}(\underline{z}) \, dN$, where dN is the Lebesgue measure on N. It should also be noted that the vector $\underline{n}(\underline{z})$ need not in general satisfy the condition $n(\underline{z})^2 = -1$. For instance, for each $\underline{z} \in N$, the real, $(n-1)$-dimensional subspace of $\mathbb{C}^n$ spanned by $e_1 + ie_2$, $e_3, \ldots, e_n$, we have that $\underline{n}(\underline{z}) = (-i/\sqrt{2})(e_1 + ie_2)$.

The restriction in Theorem 8 to the cases where n is even is not really necessary. In [13] we note that if M is as a manifold of type one, then in all dimensions $G_n^+(\underline{z} - \underline{z}')$ is a well-defined function for each $\underline{z} \in \mathring{M}$ and each $\underline{z}' \in \partial M$. Consequently, if we have a manifold M of type one lying in $\mathbb{C}^n$, with n odd and a left-regular function $f: \Omega \to A_n(\mathbb{C})$ and with $M \subseteq \Omega$, then for each $z_0 \in \mathring{M}$ we have:

$$f(\underline{z}_0) = \frac{1}{\omega} \int_{\partial M} G_n^+(\underline{z} - \underline{z}_0) \, W\underline{z} \, f(\underline{z}). \tag{6}$$

Moreover, [13], the formula (6) gives rise to a left-regular function defined on a Riemann surface covering M^+.

We now turn to look at L^p-spaces over manifolds of type one.

DEFINITION 10. Suppose M is a manifold of type one and $g: M \to A_n(\mathbb{C})$ is a measurable function with respect to Lebesgue measure over M, then g is said to be L^p-integrable, for some $p \in (0, +\infty]$ if

$$\left(\int_M \|g(\underline{z})\|^p \, dM\right)^{1/p} < +\infty,$$

where dM denotes the Lebesgue measure of M.

We denote the $A_n(\mathbb{C})$-module of $A_n(\mathbb{C})$-valued L^p-functions on M by $L^p(M, A_n(\mathbb{C}))$. We define the (left) Bergman p-space over M to be the right $A_n(\mathbb{C})$-module

$$B_l^p(M, A_n(\mathbb{C})) = \{f: M \to A_n(\mathbb{C}): f \in L^p(M, A_n(\mathbb{C})) \text{ and } f \text{ is the restriction to } \mathring{M} \text{ of a left-regular function}\}.$$

We would like to show that $B_l^p(M, A_n(\mathbb{C}))$ is a complete submodule of $L^p(M, A_n(\mathbb{C}))$, for $1 \leqslant p < \infty$. First, we may note that for each manifold M of type one and for each point $\underline{z}_0 \in \mathring{M}$, there is a real, positive number $r(\underline{z})$ and a C^1-homotopy $H_{\underline{z}_0}: B(\underline{z}_0, r(\underline{z}_0)) \cap TM_{\underline{z}_0} \times [0, 1] \to \mathbb{C}^n$ satisfying

(i) $H_{\underline{z}_0}(\underline{z}, 0) = \underline{z}$

(ii) $H_{\underline{z}_0}: B(\underline{z}_0, r(\underline{z}_0)) \cap TM_{\underline{z}_0} \times \{t\} \to \mathbb{C}^n$ is a diffeomorphism for each $t \in [0, 1]$

(iii) $H_{\underline{z}_0}(\underline{z}_0, t) = \underline{z}_0$ and $TH_{\underline{z}_0}(B(\underline{z}_0, r(\underline{z}_0)) \cap TM_{\underline{z}_0}), 1)_{\underline{z}_0} = TM_{\underline{z}_0}$ for each $t \in [0, 1]$

(iv) $H_{\underline{z}_0}(B(\underline{z}_0, r(\underline{z}_0)), \{1\}) \subseteq M$.

Here, $B(\underline{z}_0, r(\underline{z}_0))$ is the set $\{\underline{z} \in \mathbb{C}^n: \|\underline{z} - \underline{z}_0\| < r(\underline{z}_0)\}$. So, in fact, the homotopy $H_{\underline{z}_0}$ is a C^1-deformation of $B(\underline{z}_0, r(\underline{z}_0)) \cap TM_{\underline{z}_0}$ into a neighborhood of $\underline{z}_0$ within M.

It is straightforward to deduce from the existence of the homotopy $H_{\underline{z}_0}$ that there is a positive real number $r'(\underline{z}_0) \leqslant r(\underline{z}_0)$ and a C^1-homotopy

$$H'_{\underline{z}_0}: B(\underline{z}_0, r'(\underline{z}_0)) \cap TM_{\underline{z}_0} \times [0, 1] \to \mathbb{C}^n$$

satisfying conditions (i)–(iv), and further we have that

$$\|H'_{\underline{z}_0}(\underline{z}, t) - \underline{z}_0\| = \|\underline{z} - \underline{z}_0\|$$

for all $\underline{z} \in B(\underline{z}_0, r'(\underline{z}_0)) \cap TM_{\underline{z}_0}$. So, the homotopy $H_{\underline{z}_0}$ can be replaced by an isometry.

Now if $f: M^+ \to A_n(\mathbb{C})$ is a left-regular function, we have, using the function (5) and from Theorem 8 that for $r < r'(\underline{z}_0)$

$$f(\underline{z}_0) = \frac{1}{\omega_n} \int_{S(\underline{z}_0, r) \cap M} G(\underline{z} - \underline{z}_0)\, \underline{n}(\underline{z}) f(\underline{z})\, d(S(\underline{z}_0, r) \cap M), \tag{7}$$

where $S(\underline{z}_0, r) = \{\underline{z} \in \mathbb{C}^n: \|\underline{z} - \underline{z}_0\| = r\}$ and $d(S(\underline{z}_0, r) \cap M)$ is the Lebesgue measure on $S(\underline{z}_0, r) \cap M$. As $A_n(\mathbb{C})$ is a Banach algebra, it follows from (7) that there is a real positive constant C such that

$$\|f(\underline{z}_0)\| \leqslant C \int_{S(\underline{z}_0, r) \cap M} \|G(\underline{z} - \underline{z}_0)\|\ \|\underline{n}(\underline{z})\|\ \|f(\underline{z})\|\ d(S(\underline{z}_0, r) \cap M).$$

From the homotopy $H'_{\underline{z}_0}$ we have that

$$\begin{aligned}
&\int_{S(\underline{z}_0, r) \cap M} \|G(\underline{z} - \underline{z}_0)\|\ \|\underline{n}(\underline{z})\|\ \|f(\underline{z})\|\ d(S(\underline{z}_0, r) \cap M) \\
&\quad = \int_{S(\underline{z}_0, r) \cap TM_{\underline{z}_0}} \|G(H'_{\underline{z}_0}(\underline{z}', 1) - \underline{z}_0)\|\ \|\underline{n}(H'_{\underline{z}_0}(\underline{z}', 1))\| \\
&\qquad \times \|f(H'_{\underline{z}_0}(\underline{z}', 1))\|\ |J(H'_{\underline{z}_0}|_{S(\underline{z}_0, r)}(\underline{z}', 1))|\ d(S(\underline{z}_0, r) \cap TM_{\underline{z}_0}),
\end{aligned} \tag{8}$$

where $H'_{\underline{z}_0}(\underline{z}', 1) = \underline{z}$, and $J(H'_{\underline{z}_0}|_{S(\underline{z}_0, r)})$ is the Jacobian of $H'_{\underline{z}_0}|_{S(\underline{z}_0, r)}$, the restriction of $H'_{\underline{z}_0}$ to $S(\underline{z}_0, r)$. As M is a manifold of type one, then, [11], there is a constant $C_M \in R^+$ such that

$$\|G(\underline{z} - \underline{z}')\| \leqslant C_M \frac{1}{\|\underline{z} - \underline{z}'\|^{n-1}} \tag{9}$$

for each pair $\underline{z}, \underline{z}' \in M$, with $\underline{z} \neq \underline{z}'$. Consequently, we have from (8) and (9) that

$$\begin{aligned}
\|f(\underline{z}_0)\| \leqslant C \cdot C_M \int_{S(\underline{z}_0, r) \cap TM_{\underline{z}_0}} &\frac{1}{r^{n-1}} \|\underline{n}(H'_{\underline{z}_0}(\underline{z}', 1))\|\ \|f(H'_{\underline{z}_0}(\underline{z}', 1))\| \\
&\times |J(H'_{\underline{z}_0}|_{S(\underline{z}_0, r)}(\underline{z}', 1)|\ d(S(\underline{z}_0, r) \cap TM(\underline{z}_0)).
\end{aligned}$$

So, for $0 < r_1 < r_2 < r'(\underline{z}_0)$ we have

$$\begin{aligned}
\|f(\underline{z}_0)\| \leqslant \frac{C_1}{r_2 - r_1} \int_{B(\underline{z}_0, r_1, r_2) \cap TM_{\underline{z}_0}} &\frac{1}{r^{n-1}} \|\underline{n}(H'_{\underline{z}_0}(\underline{z}', 1))\| \\
&\times \|f(H'_{\underline{z}_0}(\underline{z}', 1))\|\ |J(H'_{\underline{z}_0}|_{S(\underline{z}_0, r)}(\underline{z}', 1))|\ d(S(\underline{z}_0, r) \cap TM(\underline{z}_0))\, dr,
\end{aligned}$$

where $C_1 = C \cdot C_M$, and $B(\underline{z}_0, r_1, r_2) = \{\underline{z} \in \mathbb{C}^n : r_1 < \|\underline{z} - \underline{z}_0\| < r_2\}$. Consequently,

$$\|f(\underline{z}_0)\| \leqslant \frac{C_1}{r_2 - r_1} \int_{B(\underline{z}_0, r_1, r_2) \cap M} \frac{1}{\|\underline{z} - \underline{z}_0\|^{n-1}} \|\underline{n}(\underline{z})\| \; \|f(\underline{z})\| \; |\lambda(\underline{z})| \, dM,$$

where $\lambda(\underline{z}) = J(H'_{\underline{z}0} |_{s(\underline{z}_0, r)} (\underline{z}', 1))/J(H'_{\underline{z}0}(\underline{z}', 1))$, and $J(H'_{\underline{z}0}(\underline{z}', 1))$ is the Jacobian of $H'_{\underline{z}0}(\underline{z}', 1)$. As

$$H'_{\underline{z}0} : B(\underline{z}_0, r'(\underline{z}_0)) \cap TM_{\underline{z}0} \times \{1\} \to \mathbb{C}^n$$

is a diffeomorphism, we have that $JH'_{\underline{z}0}(\underline{z}', 1) \neq 0$. Also, we may choose $r'(\underline{z}_0)$ so that $1/JH'_{\underline{z}0}(\underline{z}', 1)$ is bounded on $B(\underline{z}_0, r'(\underline{z}_0)) \cap TM_{\underline{z}0}$. Consequently, there is a constant $C(\underline{z}_0) \in R^+$ such that

$$\|f(\underline{z}_0)\| \leqslant \frac{C_1 \cdot C(\underline{z}_0)}{r_2 - r_1} \int_{B(\underline{z}_0, r_1, r_2) \cap M} \frac{\|\underline{n}(\underline{z})\|}{\|\underline{z} - \underline{z}_0\|^{n-1}} \|f(\underline{z})\| \, dM.$$

It follows from elementary continuity arguments, and condition (iii) for the homotopy H, that we can choose $r'(\underline{z}_0)$ so that $C(\underline{z}_0)$ is no larger than 2. So, we have that

$$\|f(\underline{z}_0)\| \leqslant \frac{2C_1}{r_2 - r_1} \int_{B(\underline{z}_0, r_1, r_2) \cap M} \frac{\|\underline{n}(\underline{z})\|}{\|\underline{z} - \underline{z}_0\|^{n-1}} \|f(\underline{z})\| \, dM$$

for each $\underline{z}_0 \in M$.

From the tangent bundle TM we may obtain the fiber bundle $P: G_{n-1}M \to M$, where for each $\underline{z}_0 \in M$ we have that $P^{-1}(\{\underline{z}_0\})$ is the compact Grassmann space $G_{n-1}(TM_{\underline{z}0})$ of real, $(n-1)$-dimensional subspaces of $TM_{\underline{z}0}$. As M is compact and $G_{n-1}(TM_{\underline{z}0})$ is compact, it follows that $G_{n-1}M$ is compact. Also the map

$$\|\underline{n}\| : G_{n-1}M \to R^+ \cup \{0\} : \|\underline{n}\| \; Q_{\underline{z}0} = \|\underline{n}(Q_{\underline{z}0})\|$$

is continuous, where $Q_{\underline{z}0}$ is an oriented $(n-1)$-dimensional subspace of $TM_{\underline{z}0}$ and $\underline{n}(Q_{\underline{z}0}) = \underline{n}(\underline{z})$ for each $\underline{z} \in Q_{\underline{z}0}$. As $G_{n-1}M$ is compact and $\|\underline{n}\|$ is continuous, it follows that there is a constant C_M such that

$$\|f(\underline{z}_0)\| \leqslant \frac{2C_1 C_M}{r_2 - r_1} \int_{B(\underline{z}0, r_1, r_2) \cap M} \frac{\|f(\underline{z})\|}{\|\underline{z} - \underline{z}_0\|^{n-1}} \, dM \tag{10}$$

for $0 < r_1 < r_2 < r'(\underline{z}_0)$ and each $\underline{z}_0 \in \mathring{M}$.

The argument leading up to establishing the inequality (10) is a rigorization of an argument presented in [10]. Applying Hölder's inequality to the right-hand side of expression (10), we get that

$$\|f(\underline{z}_0)\|^p \leqslant \frac{(2C_1 C_M)^p}{(r_2-r_1)^p}\left(\int_{B(\underline{z}_0,\, r_1,\, r_2)\cap M} \|\underline{z}-\underline{z}_0\|^{(1-n)q}\, dM\right)^{p/q}$$
$$\times\left(\int_{B(\underline{z}_0,\, r_1,\, r_2)\cap M} \|f(\underline{z})^p\|\, dM\right),$$

for $1<p<\infty$, and $1/p+1/q=1$. Consequently,

$$\|f(\underline{z}_0)\| \leqslant \frac{2C_1 C_M}{r_2-r_1}\|f\|_{p,\,M}\left(\int_{B(\underline{z}_0,\, r_1,\, r_2)\cap M} \|\underline{z}-\underline{z}_0\|^{(1-n)q}\, dM\right)^{1/q}, \tag{11}$$

where $\|f\|_{p,\,M}=(\int_M \|f(\underline{z})\|^p\, dM)^{1/p}$. Similarly, we have that

$$\|f(\underline{z}_0)\| \leqslant \frac{2C_1 C_M}{r_2-r_1}\|f\|_{\infty,\,M}\int_{B(\underline{z}_0,\, r_1,\, r_2)\cap M} \|\underline{z}-\underline{z}_0\|^{(1-n)}\, dM, \tag{12}$$

for each $f\in B_1^{\infty}(M, A_n(\mathbb{C}))$, where $\|f\|_{\infty,\,M}=\sup_{\underline{z}\in M}\|f(z)\|$, and

$$\|f(\underline{z}_0)\| \leqslant \frac{2C_1 C_M}{r_2-r_1}\|f\|_{1,\,M}\sup_{\underline{z}\in B(\underline{z}_0,\, r_1,\, r_2)\cap M}\left(\frac{1}{\|\underline{z}-\underline{z}_0\|^{n-1}}\right) \tag{13}$$

for $f\in B_1^1(M, A_n(C))$.

Suppose now that M' is a manifold of type one lying in the interior of M. Then the collection $\{B(\underline{z}_0, r'(\underline{z}_0))\cap M\colon \underline{z}_0\in M'\}$ is an open covering of M'. Consequently, we have from Lebesgue's covering lemma that there is a positive number $r(M')$ such that $B(\underline{z}_0, r(M'))\subseteq B(\underline{z}_0, r'(\underline{z}_0))$ for each $\underline{z}_0\in M'$. So there exist numbers $r_1(M')$ and $r_2(M')$ with $0<r_1(M')<r_2(M')<r(M')$. From the inequalities (11), (12), and (13) we now have that

$$\|f(\underline{z}_0)\| \leqslant C_p(M')\,\|f\|_p, \tag{14}$$

for each $\underline{z}_0\in M'$ and each $f\in B_1^p(M, A_n(\mathbb{C}))$, where $1\leqslant p\leqslant\infty$, and $C_p(M')\in R^+$.

As the right $A_n(\mathbb{C})$-module, $M_1(M, A_n(\mathbb{C}))$, of complex left-regular functions defined on M^+ is a Fréchet space [11], it follows from (14) that we have deduced:

THEOREM 9. *For $1\leqslant p\leqslant\infty$ the right $A_n(\mathbb{C})$-module $B_1^p(M, A_n(\mathbb{C}))$ is complete.*

The special case where $M \subseteq R^n$ is described in [6, Ch. 2].

Suppose now that $B: \Omega \to \mathbb{C}$ is a holomorphic function, then we let $b(\underline{z})$ denote its holomorphic gradient $\sum_{j=1}^n e_j(\partial B/\partial z_j)(\underline{z})$.

DEFINITION 11. A holomorphic function $g: \Omega \to A_n(\mathbb{C})$ is said to be complex left-regular with respect to the potential B if

$$D_{\mathbb{C}}\, g(\underline{z}) - b(\underline{z})\, g(\underline{z}) = 0,$$

for all $\underline{z} \in \Omega$.

Suppose now that M is a manifold of type one lying in Ω. Then we have that for each $\underline{z}_0 \in \mathring{M}$

$$g(\underline{z}_0) = \frac{1}{\omega_n} \int_{\partial M} W_B^+(\underline{z}, \underline{z}_0)\, W\underline{z} g(\underline{z}), \tag{15}$$

where $W_B^+(\underline{z}, \underline{z}_0) = G^+(\underline{z} - \underline{z}_0)\, e^{B(\underline{z}_0) - B(\underline{z})}$, and g is complex left-regular with respect to B. As $G^+(\underline{z} - \underline{z}_0)\, e^{B(\underline{z}_0) - B(\underline{z})}$ is well-defined on $M^+ \cap \Omega$ when n is even, and is well-defined on a Riemann surface covering $M^+ \cap \Omega$ when n is odd, it follows from (15) that $g(\underline{z}_0)$ may be holomorphically extended to $M^+ \cup \Omega$, when n is even, and to some covering of this open set when n is odd.

This last observation on holomorphic continuation gives a distinct improvement to the holomorphic continuation results obtained in [11] for solutions to the inhomogeneous Dirac equation

$$D_{\mathbb{C}}\, g(\underline{z}) + A(\underline{z})\, g(\underline{z}) = 0,$$

for the special case where $A(\underline{z}) = D_{\mathbb{C}} B(\underline{z})$ for some $B: \Omega \to \mathbb{C}$. In [11] the results relied heavily upon certain geometric constraints, which do not arise in the context considered here. From (15) we may deduce the following result by similar means to those used to obtain the inequality (14).

LEMMA 1. *Suppose that* $g: \Omega \to A_n(\mathbb{C})$ *is complex left-regular with respect to the potential B. Suppose also that* $M \subseteq \Omega$ *is a manifold of type one. Then there is a positive constant C, and for each point* $\underline{z}_0 \in \mathring{M}$ *there is a positive number* $r'(\underline{z}_0)$, *such that*

$$\|g(\underline{z}_0)\| \leqslant \frac{C}{r_2 - r_1} \int_{B(\underline{z}_0, r_1, r_2) \cap M} \frac{\|g(\underline{z})\|}{\|\underline{z} - \underline{z}_0\|^{n-1}}\, dM, \tag{16}$$

for $0 < r_1 < r_2 < r'(\underline{z}_0)$.

If we now define the (left-) Bergman p-space over M, with respect to B, to be the right $A_n(\mathbb{C})$-module $B^p_{1,B}(M, A_n(\mathbb{C})) = \{f\colon M \to A_n(\mathbb{C})\colon f \in L^p(M, A_n(\mathbb{C}))$, and f is the restriction to M of a complex left-regular function with respect to $B\}$, then from the inequality (16) we may deduce by similar arguments to those used to establish Theorem 9:

THEOREM 10. *For $1 \leqslant p \leqslant \infty$ the right $A_n(\mathbb{C})$-module $B^p_{1,B}(M, A_n(\mathbb{C}))$ is complete.*

We conclude this section by stating the following theorem which follows automatically from [13].

THEOREM 11. *Suppose that M is a manifold of type one with a Lipschitz continuous boundary. Suppose also that $h\colon \partial M \to A_n(\mathbb{C})$ is such that $h(\underline{z})\, e^{B(\underline{z})}$ is a Hölder continuous function with exponent $\alpha \in (0, 1)$, where B is a complex-valued holomorphic function defined in a neighborhood of M. Then:*

(a) *The integral*

$$P.V. \frac{1}{\omega_n} \int_{\partial M} W_B^+(\underline{z}, \underline{z}')\, W\underline{z} h(\underline{z})$$

is bounded on the set of smooth points in ∂M, and the function

$$e^{B(\underline{z}')}\, P.V. \frac{1}{\omega} \int_{\partial M} W_B^+(\underline{z}, \underline{z}')\, W\underline{z} h(\underline{z})$$

is Hölder continuous with exponent α.

(b) *Suppose that $\underline{z}'$ is a smooth point in ∂M (i.e., $\underline{z}'$ has a tangent space within ∂M) and $\lambda_{\underline{z}'}\colon (0, 1] \to M^+$ is a smooth map satisfying the following conditions:*

(i) $\lim_{t \to 0} \lambda_{\underline{z}'}(t) = \underline{z}_0$.

(ii) *$\lambda_{\underline{z}'}$ has a smooth extension to $[0, 1]$.*

(iii) *$(d\lambda_{\underline{z}'}/dt)|_{t=0}$ is not a member of $\mathbb{C}T\,\partial M_{\underline{z}'}$, the complexification of the tangent space $T\,\partial M_{\underline{z}'}$.*

Then

$$\lim_{t \to 1} \frac{1}{\omega_n} \int_{\partial M} W_B^+(\underline{z}, \lambda_{\underline{z}'}(t))\, W\underline{z} h(\underline{z}) = \frac{1}{2} h(\underline{z}') + P.V. \frac{1}{\omega_n} \int_{\partial M} W_B^+(\underline{z}, \underline{z}')\, W\underline{z} h(\underline{z}).$$

(c)
$$\begin{aligned} &\lim_{t \to 1} (1/\omega_n) \textstyle\int_{\partial M} W_B^+(\underline{z}, 2z' - \lambda_{\underline{z}'}(t))\, W\underline{z} h(\underline{z}) \\ &\quad = -\tfrac{1}{2} h(\underline{z}') + P.V.(1/\omega_n) \textstyle\int_{\partial M} W_B^+(\underline{z}, \underline{z}')\, W\underline{z} h(\underline{z}). \end{aligned}$$

Parts (b) and (c) of the previous theorem give the Plemelj formulae associated to a complex left-regular function with respect to a potential in $\mathbb{C}^n$.

Intrinsic Dirac Operators

In [11, Proposition 2] we show that if $g\colon M \to A_n(\mathbb{C})$ is a bounded, integrable function, where M is a manifold of type one, then

$$\frac{1}{\omega_n}\int_M G^+(\underline{z}-\underline{z}_0)\, g(\underline{z})\, d\underline{z}^n \tag{17}$$

is a well-defined, bounded function on M. In fact, in [11] we assume that n is even. However, the proof given in [11] carries over with no major change to all n.

We shall denote the function given by expression (17) by $T_M g(\underline{z}_0)$. In [11] we show that if $g(\underline{z})$ extends to a holomorphic function in a neighborhood of $\mathring{M}$, then so does $T_M g(\underline{z}_0)$. In this section we shall simply assume that g is a C^1-function on $\mathring{M}$. We begin by deducing:

Theorem 12. *Suppose M is a manifold of type one, and $g\colon \mathring{M} \to A_n(\mathbb{C})$ is a bounded C^1-function. Then $T_M g(\underline{z}_0)$ is C^1 on $\mathring{M}$.*

Proof. Suppose first that $\underline{z}_0 \in \mathring{M}$ and $\phi\colon (-\frac{1}{2}, \frac{1}{2}) \to M$ is a C^1-map, with $\phi(0)=\underline{z}_0$. In order to show that $T_M g(\underline{z}_0)$ is differentiable on $\mathring{M}$, we shall first show that

$$\lim_{t\to 0} \operatorname{sgn}(t)\,\frac{(T_M g(\phi(0)) - T_M g(\phi(t)))}{\|\phi(0)-\phi(t)\|}$$

exists, where $\operatorname{sgn}(t)=1$ if $t>0$, and $\operatorname{sgn}(t)=-1$ if $t<0$. Suppose that U is an open subset of M, and that $z_0 \in U$. As $G^+(\underline{z}-\underline{z}_0)$ is a holomorphic function on $\mathbb{C}^n \backslash N(z_0)$, or a double covering of this domain, it follows that the function

$$\frac{1}{\omega_n}\int_{M\backslash C} G^+(\underline{z}-\underline{z}_0')\, g(\underline{z})\, d\underline{z}^n$$

is differentiable at $\underline{z}_0$ whenever C is a measurable subset of M and $U \subseteq C$. It follows that we need only determine

$$\lim_{t\to 0} \frac{\operatorname{sgn}(t)}{\|\phi(0)-\phi(t)\|}\,\frac{1}{\omega_n}\int_C (G^+(\underline{z}-\phi(0)) - G^+(\underline{z}-\phi(t)))\, g(\underline{z})\, d\underline{z}^n.$$

Now consider a C^1-diffeomorphism

$$E\colon (-\tfrac{1}{2}, \tfrac{1}{2})^n \to M,$$

where $(-\frac{1}{2}, \frac{1}{2})^n = \{(\lambda_1, ..., \lambda_n)\colon \lambda_1, ..., \lambda_n \in (-\frac{1}{2}, \frac{1}{2})\}$. We may choose this diffeomorphism so that for some $\lambda_1, ..., \lambda_{n-1} \in (-\frac{1}{2}, \frac{1}{2})$ we have $E(\lambda_1, ..., \lambda_{n-1}, t) = \phi(t)$. We may place $C = E([-\frac{1}{3}, \frac{1}{3}]^n)$, where $[-\frac{1}{3}, \frac{1}{3}]^n = \{(\lambda_1, ..., \lambda_n)\colon \lambda_1, ..., \lambda_n \in [-\frac{1}{3}, \frac{1}{3}]\}$. Now for each $\phi(t) \in E([-\frac{1}{3}, \frac{1}{3}]^n)$ and each $\underline{z} \in E([-\frac{1}{3}, \frac{1}{3}]^n)$ we may find a vector $\lambda(t, \underline{z}) \in \mathbb{C}^n$ such that

(i) $\|\phi(0) - \phi(t)\| = \|\lambda(t, \underline{z})\|$

(ii) $\underline{z} - \lambda(t, \underline{z}) \in E((-\frac{1}{2}, \frac{1}{2})^n)$

(iii) $\underline{z} - \lambda(t, \underline{z}) = E(\lambda_1(\underline{z}), ..., \lambda_{n-1}(\underline{z}), t)$

for some $\lambda_1(\underline{z}), ..., \lambda(\underline{z}) \in [-\frac{1}{3}, \frac{1}{3}]$.

Moreover, C can be chosen so that

$$\|\lambda(t, \underline{z}) - \phi(t) + \phi(0)\| \leqslant C' \ \|\phi(t) - \phi(0)\|^{1+\varepsilon} \tag{18}$$

for some $C', \varepsilon \in R^+$.

From now on, we shall express $\phi(t)$ as $\underline{z}_0 + \psi(t)$. So $\psi(t) = \phi(t) - \phi(0)$. We can now rewrite the expression

$$\frac{1}{\|\phi(0) - \phi(t)\|} \frac{1}{\omega_n} \int_C (G^+(\underline{z} - \phi(0)) - G^+(\underline{z} - \phi(t)))\, g(\underline{z})\, d\underline{z}^n$$

as

$$\frac{1}{\omega_n \|\psi(t)\|} \int_C (G^+(\underline{z} - \underline{z}_0) - G^+(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z}))$$
$$+ G^+(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z}) - G^+(\underline{z} - \underline{z}_0 - \psi(t)))\, g(\underline{z})\, d\underline{z}^n.$$

Now

$$\frac{1}{\omega_n \|\psi(t)\|} \int_C (G^+(\underline{z} - \underline{z}_0) - G^+(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z})))\, g(\underline{z})\, d\underline{z}^n$$
$$= -\frac{1}{\omega_n \|\psi(t)\|} \int_{C_1(t)} G^+(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z}))\, g(\underline{z})\, d\underline{z}^n$$
$$+ \frac{1}{\omega_n \|\psi(t)\|} \int_{C_2(t)} G^+(\underline{z} - \underline{z}_0)(g(\underline{z}) - g(\underline{z} - \lambda(t, \underline{z})))\, d\underline{z}^n$$
$$+ \frac{1}{\omega_n \|\psi(t)\|} \int_{C_3(t)} G^+(\underline{z} - \underline{z}_0)\, g(\underline{z})\, d\underline{z}^n,$$

where

$$C_1(t)=\{\underline{z}\in C\colon \inf_{\underline{z}'\in E([-1/3,\,1/3]^{n-1}\times\{0\})}\|\underline{z}-\underline{z}'\|\leqslant\|\psi(t)\|\},$$

$$C_3(t)=\{\underline{z}\in C\colon \inf_{\underline{z}'\in E([-1/3,\,1/3]^{n-1}\times\{1\})}\|\underline{z}-\underline{z}'\|\leqslant\|\psi(t)\|\},$$

and

$$C_2(t)=C\backslash(C_1(t)\cup C_3(t)).$$

It follows that

$$\lim_{t\to 0}\frac{\operatorname{sgn}(t)}{\omega_n\,\|\psi(t)\|}\left(-\int_{C_1(t)} G^+(\underline{z}-\underline{z}_0-\lambda(t,\underline{z}))\,g(\underline{z})\,d\underline{z}^n\right.$$
$$\left.+\int_{C_2(t)} G^+(\underline{z}-\underline{z}_0)\,g(\underline{z})\,d\underline{z}^n\right)$$
$$=\frac{1}{\omega_n}\int_{E([-1/3,\,1/3]^{n-1}\times\{0\})\cup E([-1/3,\,1/3]^{n-1}\times\{1\})}$$
$$\times\sum_{j=1} G^+(\underline{z}-\underline{z}_0)\,g(\underline{z})\,\lambda_j(\underline{z})\,d\hat{z}_j$$

where $\lambda_j(\underline{z})\in S^1=\{z\in\mathbb{C}\colon |z|=1\}$. Moreover, it may be observed that the function $\lambda_j(\underline{z})$ is continuous on $E([-\frac{1}{3},\frac{1}{3}]^{n-1}\times\{0\})\cup E([-\frac{1}{3},\frac{1}{3}]^{n-1}\times\{1\})$. We also have that as g is a C^1-function, then

$$\lim_{t\to 0}\frac{\operatorname{sgn}(t)}{\omega_n\,\|\psi(t)\|}\int_{C_2(t)} G^+(\underline{z}-\underline{z}_0)(g(\underline{z})-g(\underline{z}-\lambda(t,\underline{z})))\,d\underline{z}^n$$
$$=\frac{1}{\omega_n}\int_C G^+(\underline{z}-\underline{z}_0)\,g_{\lambda(0,\underline{z})}(\underline{z})\,d\underline{z}^n, \tag{19}$$

where $g_{\lambda(0,\underline{z})}(\underline{z})=\lim_{t\to 0}\operatorname{sgn}(t)(g(\underline{z})-g(\underline{z}-\lambda(t,\underline{z})))/\|\psi(t)\|$. As g is a bounded, C^1-function on C, it follows that $g_{\lambda(0,\underline{z})}$ is a bounded, C^0-function on C. So from [11, Proposition 2] we have that the right-hand side of expression (19) is well-defined.

We now turn to look at

$$\frac{1}{\omega_n\,\|\psi(t)\|}\int_C (G^+(\underline{z}-\underline{z}_0-\lambda(t,\underline{z}))-G^+(\underline{z}-\underline{z}_0-\psi(t)))\,g(\underline{z})\,d\underline{z}^n.$$

First, we note, [13], that there is a constant $C(M)\in R^+$ such that for each $\underline{\omega},\underline{\omega}'\in M$ with $\underline{\omega}\neq\underline{\omega}'$, we have

$$\|G^+(\underline{z}-\underline{\omega})-G^+(\underline{z}-\underline{\omega}')\|\leqslant C(M)\,\|\underline{\omega}-\underline{\omega}'\|\,\frac{(\sum_{j=0}^n\|\underline{z}-\underline{\omega}\|^{n-j}\,\|\underline{z}-\underline{\omega}'\|^{j})}{|(\underline{z}-\underline{\omega})^2|^{n/2}\,|(\underline{z}-\underline{\omega}')^2|^{n/2}} \tag{20}$$

for all $\underline{z} \in M \setminus \{\underline{\omega}, \underline{\omega}'\}$. For $d \in R^+$ let us place

$$D_1(t, d) = \{\underline{\omega} \in \mathbb{C}^n \colon \|\underline{\omega} - \underline{z}_0 - \psi(t)\| < d\} \cap C$$

and

$$D_2(d) = \{\underline{\omega} \in \mathbb{C}^n \colon \|\omega - z_0\| < d\} \cap C.$$

Then, from the inequality (20) we have for some $C_1(M) \in R^+$ that

$$\frac{1}{\omega_n} \left\| \int_{C \setminus (D_1(t, d) \cup D_2(d))} (G^+(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z})) - G^+(\underline{z} - \underline{z}_0 - \psi(t)))\, g(\underline{z})\, d\underline{z}^n \right\|$$

$$\leqslant C_1(M) \sup_{\underline{z} \in C} \int_{C \setminus (D_1(t, d) \cup D_2(d))} \|\lambda(t, \underline{z}) - \psi(t)\|$$

$$\times \frac{(\sum_{j=1}^n \|\underline{z} - \underline{z}_0 - \lambda(t, \underline{z})\|^{n-j} \|\underline{z} - \underline{z}_0 - \psi(t)\|^j)}{|(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z}))^2|^{n/2} |\underline{z} - \underline{z}_0 - \psi(t))^2|^{n/2}}\, dM.$$

From (18) and elementary continuity arguments is now follows that there exists $\alpha(d)$ and $C(M, g) \in R^+$ such that:

(a) $\alpha(d)\, |(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z}))^2|^{1/2} \geqslant |(\underline{z} - \underline{z}_0 - \psi(t))^2|^{1/2}$ for each $\underline{z} \in C \setminus (D_1(t, d) \cup D_2(d))$, and

(b)

$$\frac{1}{\omega_n \|\psi(t)\|} \left\| \int_{C \setminus (D_1(t, d) \cup D_2(d))} (G^+(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z})) \right.$$

$$\left. - G^+(\underline{z} - \underline{z}_0 - \psi(t)))\, g(\underline{z})\, d\underline{z}^n \right\|$$

$$\leqslant \frac{C(M, g)}{\omega_n} \|\psi(t)\|^\varepsilon$$

$$\times \int_{C \setminus (D_1(t, d) \cup D_2(d))} \frac{\sum_{j=0}^n \|\underline{z} - \underline{z}_0 - \lambda(t, \underline{z})\|^{n-j} \|\underline{z} - \underline{z}_0 - \psi(t)\|^j}{\|\underline{z} - \underline{z}_0 - \psi(t)\|^{2n}\, \alpha(d)^n}\, dM.$$

Elementary continuity considerations also tell us that there is a function $\beta(d) \in R^+$ such that

$$|(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z}))^2|^{1/2} \leqslant \beta(d)\, |(\underline{z} - \underline{z}_0 - \psi(t))^2|^{1/2}.$$

Moreover, elementary geometric considerations now tell us that if $d \geqslant 4C \|\psi(t)\|^{1+\varepsilon}$, then there are constants $\alpha, \beta \in R^+$ such that

$$|(\underline{z} - \underline{z}_0 - \psi(t))^2|^{1/2} \leqslant \alpha\, |(\underline{z} - \underline{z}_0 - \lambda(t, \underline{z}))^2|^{1/2}$$

and

$$|(\underline{z}-\underline{z}_0-\lambda(t,\underline{z}))^2|^{1/2}\leqslant\beta\,|(\underline{z}-\underline{z}_0-\psi(t))^2|^{1/2}$$

for all $\underline{z}\in C\backslash(D_1(t,d)\cup D_2(d))$. Consequently, there is a positive number $C_1(M,g)$ such that

$$\frac{1}{\omega_n\,\|\psi(t)\|}\left\|\int_{C\backslash(D_1(t,d)\cup D_2((d))}(G^+(\underline{z}-\underline{z}_0-\lambda(t,\underline{z}))\right.$$
$$\left.-G^+(\underline{z}-\underline{z}_0-\psi(t)))\,g(\underline{z})\,d\underline{z}^n\right\|$$
$$\leqslant\frac{C_1(M,g)}{\omega_n}\,\|\psi(t)\|^{\varepsilon}\int_{C\backslash(D_1(t,d)\cup D_2(d))}\frac{1}{\|\underline{z}-\underline{z}_0-\psi(t)\|^n}\,dM,$$

for $d\geqslant 4C\,\|\psi(t)\|^{1+\varepsilon}$.

Again, elementary continuity arguments and standard inequalities give us that

$$\int_{C\backslash(D_1(t,d)\cup D_2(d))}\frac{1}{\|\underline{z}-\underline{z}_0-\psi(t)\|^n}\,dM\leqslant K\log d,$$

for some constant $K\in R^+$. Consequently,

$$\frac{1}{\omega_n\,\|\psi(t)\|}\left\|\int_{C\backslash(D_1(t,d)\cup D_2(d))}(G^+(\underline{z}-\underline{z}_0-\lambda(t,\underline{z}))\right.$$
$$\left.-G^+(\underline{z}-\underline{z}_0-\psi(t)))\,g(\underline{z})\,d\underline{z}^n\right\|$$
$$\leqslant\frac{KC_1(M,g)}{\omega_n}\,\|\psi(t)\|^{\varepsilon}\log 4C\,\|\psi(t)\|^{1+\varepsilon},$$

for $d=4C\,\|\psi(t)\|^{1+\varepsilon}$. Moreover,

$$\lim_{t\to 0}\,\|\psi(t)\|^{\varepsilon}\log 4C\,\|\psi(t)\|^{1+\varepsilon}=0.$$

We may now observe that

$$\frac{1}{\omega_n\,\|\psi(t)\|}\left\|\int_{D_1(t,d)\cup D_2(d)}(G^+(\underline{z}-\underline{z}_0-\lambda(t,\underline{z}))-G^+(\underline{z}-\underline{z}_0-\psi(t)))\,g(\underline{z})\,d\underline{z}^n\right\|$$
$$\leqslant\frac{C'}{\omega_n\,\|\psi(t)\|}\left(\int_{D_1(t,16d)}\|G^+(\underline{z}-\underline{z}_0-\psi(t))\|\,\|g(\underline{z})\|\,dM\right.$$
$$\left.+\int_{D_2(16d)}\|G^+(\underline{z}-\underline{z}_0-\lambda(t,\underline{z}))\|\,\|g(\underline{z})\|\,dM\right),$$

where $C' \in R^+$ and $d = 4C\,\|\psi(t)\|^{1+\varepsilon}$. In turn, this expression is bounded above by

$$\frac{2C''}{\omega_n} \cdot \frac{1}{\|\psi(t)\|} \sup_{\underline{z} \in C} \|g(\underline{z})\|\; 64C\,\|\psi(t)\|^{1+\varepsilon},$$

for some $C'' \in R^+$. This expression tends to zero as t tends to zero. So, we have shown that $\lim_{t\to 0} \operatorname{sgn}(t)((T_M g(\phi(0)) - T_M g(\phi(t)))/\|\phi(0) - \phi(t)\|)$ exists and is equal to

$$\frac{1}{\omega_n}\int_{M\setminus C}\left(\lim_{t\to 0} \operatorname{sgn}(t)\,\frac{(G^+(\underline{z}-\underline{z}_0-\psi(t)) - G^+(\underline{z}-\underline{z}_0))}{\|\psi(t)\|}\right) g(\underline{z})\, d\underline{z}^n$$
$$+\frac{1}{\omega_n}\int_{E([-1/3,\,1/3]^{n-1}\times\{0\})\cup E([-1/3,\,1/3]^{n-1}\times\{1\})}$$
$$\times \sum_{j=1}^{n} G^+(\underline{z}-\underline{z}_0)\, g(\underline{z})\, \lambda_j(\underline{z})\, d\hat{z}_j$$
$$+\frac{1}{\omega_n}\int_C G^+(\underline{z}-\underline{z}_0)\, g_{\lambda(0,\,\underline{z})}(\underline{z})\, d\underline{z}^n. \tag{21}$$

The first two terms in expression (21) are restrictions to C of holomorphic functions on C^+. So in order to show that the term appearing on the left-hand side of (21) is a continuous function, it is enough to show that the integral

$$\frac{1}{\omega_n}\int_C G^+(\underline{z}-\underline{z}_0)\, g_{\lambda(0,\,\underline{z})}(\underline{z})\, d\underline{z}^n$$

defines a continuous function on $\mathring{C}$. To do this, consider a point $\underline{z}_1 \in \mathring{C}\setminus\{\underline{z}\}$ and note that

$$\left\|\frac{1}{\omega_n}\int_C (G^+(\underline{z}-\underline{z}_0) - G^+(\underline{z}-\underline{z}_1))\, g_{\lambda(0,\,\underline{z})}(\underline{z})\, d\underline{z}^n\right\|$$
$$\leqslant C' \sup_{\underline{z}\in C} \|g_{\lambda(0,\,\underline{z})}(\underline{z})\| \int_C \|G^+(\underline{z}-\underline{z}_0) - G^+(\underline{z}-\underline{z}_1)\|\, dM,$$

for some constant C'. Using the inequality (20) it may be deduced, by similar arguments to those used earlier in this proof, that

$$\int_{C\setminus(B(\underline{z}_0,\,4\,\|\underline{z}_0-\underline{z}_1\|)\cup B(\underline{z}_1,\,4\,\|\underline{z}_0-\underline{z}_1\|))\cap M} \|G^+(\underline{z}-\underline{z}_0) - G^+(\underline{z}-\underline{z}_1)\|\, dM$$
$$\leqslant C(M)\,\|\underline{z}_0-\underline{z}_1\| \log 4\,\|\underline{z}_0-\underline{z}_1\|,$$

for some constant $C(M)$. Moreover,

$$\lim_{\underline{z}_1 \to \underline{z}_0} \|\underline{z}_0 - \underline{z}_1\| \log 4 \|\underline{z}_0 - \underline{z}_1\| = 0.$$

Also, by similar arguments to those used earlier in the proof, we have that

$$\int_{(B(\underline{z}_0, 4\|\underline{z}_0 - \underline{z}_1\|) \cup B(\underline{z}_1, 4\|\underline{z}_0 - \underline{z}_1\|)} \|G^+(\underline{z} - \underline{z}_0) - G^+(\underline{z} - \underline{z}_1)\| \, dM \leqslant C \|\underline{z}_0 - \underline{z}_1\|$$

for some $C \in R^+$. It follows that $1/\omega_n \int_C G^+(\underline{z} - \underline{z}_0) \, g_{\lambda(0,\underline{z})}(\underline{z}) \, d\underline{z}^n$ defines a continuous function on C, so on varying $\underline{z}_0$ and $\phi(t)$ we have that

$$\lim_{t \to 0} \operatorname{sgn}(t) \frac{(T_M g(\phi(0)) - T_M g(\phi(t)))}{\|\phi(0) - \phi(t)\|}$$

defines a continuous function on $\mathring{M}$. It now follows that $T_M g(\underline{z}_0)$ is a C^1-function on $\mathring{M}$. ∎

For each set $C, \subseteq \mathring{M}$, satisfying the conditions appearing in the proof of Theorem 12, we have that

$$\left\| \frac{1}{\omega_n} \int_{M \setminus C} G^+(\underline{z} - \underline{z}_0) \, g(\underline{z}) \, d\underline{z}^n \right\| \leqslant C_1 \sup_{\underline{z} \in M} \|g(\underline{z})\| \sup_{\underline{z}_1 \in M} \int_M \|G^+(\underline{z} - \underline{z}_0)\| \, dM,$$

for some constant $C_1 \in R^+$. We also have that

$$\left\| \frac{1}{\omega_n} \int_C G^+(\underline{z} - \underline{z}_0) \, g_{\lambda(0,\underline{z})}(\underline{z}) \, d\underline{z}^n \right\|$$

$$\leqslant C_2 \sup_{\underline{z} \in M} \|Dg(\underline{z})\| \sup_{\underline{z}_1 \in M} \int_M \|G^+(\underline{z} - \underline{z}_0)\| \, dM,$$

for some constant $C_2 \in R^+$. Here, $Dg(\underline{z})$ denotes the derivative of g. It follows from the homogeneity of $G^+(\underline{z} - \underline{z}_0)$ that there is a constant $C_3 \in R^+$ such that for each $C \subseteq M$ satisfying the conditions appearing in the proof of Theorem 12 we have

$$\left\| \frac{1}{\omega_n} \int_{E([-1/3, 1/3]^{n-1} \times \{0\}) \cup E([-1/3, 1/3]^{n-1} \times \{1\})} G^+(\underline{z} - \underline{z}_0) \, g(\underline{z}) \, d\underline{z}^n \right\|$$

$$\leqslant C_3 \sup_{\underline{z} \in M} \|g(\underline{z})\|.$$

Consequently, we have:

PROPOSITION 4. *Suppose* $g: M \to A_n(\mathbb{C})$ *is a* C^1*-function, and* $\sup_{\underline{z} \in \mathring{M}} \|Dg(\underline{z})\| < \infty$. *Then* $DT_M g(\underline{z})$ *is a bounded function on* $\mathring{M}$.

COROLLARY 1. *Suppose* $g: M \to A_n(\mathbb{C})$ *is a* C^r*-function for some* $r \in \mathbb{N}$, *and each derivative of* g *is bounded on* $\mathring{M}$. *Then* $T_M g(\underline{z})$ *is a* C^r*-function on* M *and each derivative of* $T_M g$ *is bounded on* $\mathring{M}$.

COROLLARY 2. *Suppose* $g: M \to A_n(\mathbb{C})$ *is a* C^∞*-function, and each derivative of* g *is bounded on* $\mathring{M}$. *Then* $T_M g(\underline{z})$ *is a* C^∞*-function on* M *and each derivative of* $T_M g$ *is bounded on* $\mathring{M}$.

When $\mathring{M}$ is a domain lying in a real, n-dimensional vector subspace of $\mathbb{C}^n$, then Theorem 12, Proposition 4, and Corollaries 1 and 2 can all be deduced by direct analogues of the proofs appearing in [6, 7], and elsewhere.

DEFINITION 12. Suppose that M is a manifold of type one. Then M is called a simple manifold of type one if for each $\underline{z}_0 \in \mathring{M}$ there are C^1-functions $\phi_{j, \underline{z}_0}: (-\frac{1}{2}, \frac{1}{2}) \to M$ for $j = 1, \ldots, n$, such that

(i) $\phi_{j, \underline{z}_0}(0) = \underline{z}_0$,

(ii) $\phi_{j, \underline{z}_0}(t) = \underline{z}_0 + \lambda_j(t)\, e_j$,

where $\lambda_j(t) \in \mathbb{C}$.

Clearly, any manifold of type one lying in R^n is a simple manifold of type one. Also, the set $\{\lambda_1 z(\lambda_1)\, e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n\colon 0 \leqslant \lambda_j \leqslant 1$ for $1 \leqslant j \leqslant n$, and $z(\lambda_1) \in \mathbb{C}$ with $\operatorname{Re} z(\lambda_1) \neq 0\}$ is a simple manifold of type one. Elementary constructions of simple examples like the previous one abound, so it is easily seen that there are many examples of these kinds of manifolds in $\mathbb{C}^n$.

Suppose now that M is a simple manifold of type one, and $g: \mathring{M} \to A_n(\mathbb{C})$ is a C^1-function. Then from Theorem 12 we have that

$$-\sum_{j=1}^{n} e_j \lim_{t \to 0} \frac{g(\underline{z}_0) - g(\phi_{\underline{z}_0}(t))}{\lambda_j(t)}$$

is well-defined, and gives a C^0-function on $\mathring{M}$. For M, a manifold of type one, let $C^1(\mathring{M}, A_n(\mathbb{C}))$ denote the $A_n(\mathbb{C})$-module of $A_n(\mathbb{C})$-valued C^1-functions on $\mathring{M}$, and $C^0(\mathring{M}, A_n(\mathbb{C}))$ denote the $A_n(\mathbb{C})$-module of $A_n(\mathbb{C})$-valued C^1-functions on $\mathring{M}$.

We are now ready to introduce intrinsic Dirac operators over simple manifolds of type one.

DEFINITION 13. Given a simple manifold M of type one, the intrinsic Dirac operator on M is defined to be the operator

$$D_M: C^1(M, A_n(\mathbb{C}) \to C^0(M, A_n(\mathbb{C})): g(\underline{z}_0)$$

$$\mapsto \sum_{j=1}^{n} e_j \lim_{t \to 0} \frac{(g(\underline{z}_0) - g(\phi_{\underline{z}_0}(t)))}{\lambda_j(t)}.$$

If $g(\underline{z})$ is the restriction to M of some holomorphic function $g^+(\underline{z})$, then it may easily be observed that

$$D_M g(\underline{z}) = \sum_{j=1}^{n} e_j \frac{\partial g^+}{\partial z_j}(\underline{z})\bigg|_{\mathring{M}}. \tag{22}$$

Using Theorem 12, Proposition 4, and Corollary 1 we may now deduce:

THEOREM 13. *Suppose that M is a simple manifold of type one, and $g: \mathring{M} \to A_n(\mathbb{C})$ is a bounded C^1-function with a bounded derivative. Then for each $\underline{z}_0 \in \mathring{M}$ we have*

$$D_M T_M g(\underline{z}_0) = -g(\underline{z}_0).$$

Proof. First, it may be observed that the integral

$$\frac{1}{\omega_n} \int_{M \setminus C} G^+(\underline{z} - \underline{z}_0)\, g(\underline{z})\, d\underline{z}^n$$

defines a complex left-regular function on the domain C^+, where C is the set introduced in the proof of Theorem 12. It now follows from expression (22) that

$$D_M T_M g(\underline{z}_0) = D_M \frac{1}{\omega_n} \int_C G^+(\underline{z} - \underline{z}_0)\, g(\underline{z})\, d\underline{z}^n.$$

From the proof of Theorem 12 we have that

$$D_M \frac{1}{\omega_n} \int_C G^+(\underline{z} - \underline{z}_0)\, g(\underline{z})\, d\underline{z}^n$$

$$= \frac{1}{\omega_n} \int_{\bar{C} \setminus C} W\underline{z}\mu(\underline{z})\, G^+(\underline{z} - \underline{z}_0)\, g(\underline{z}) - \frac{1}{\omega_n} \int_C \gamma(\underline{z})\, D_M G^+(\underline{z} - \underline{z}_0)\, g(\underline{z})\, d\underline{z}^n,$$

where $\mu(\underline{z})$ and $\gamma(\underline{z}) \in S^1 \subseteq \mathbb{C}$. Moreover, $\gamma(\underline{z})$ tends to 1 as $\underline{z}$ tends to $\underline{z}_0$, and $\mu(\underline{z})$ tends to 1 as C shrinks around the point $\underline{z}_0$. Consequently, for each $\varepsilon > 0$ we may choose C so that

$$\left\| \frac{1}{\omega_n} \int_{\bar{C} \setminus C} W\underline{z}(\mu(\underline{z}) - 1)\, G^+(\underline{z} - \underline{z}_0)\, g(\underline{z}) \right\| < \frac{\varepsilon}{2}$$

and

$$\left\| \frac{1}{\omega_n} \int_C (\gamma(\underline{z}) - 1)\, D_M G^+(\underline{z} - \underline{z}_0)\, g(\underline{z})\, d\underline{z}^n \right\| < \frac{\varepsilon}{2}.$$

An elementary calculation gives us that

$$g(\underline{z}_0) = \frac{1}{\omega_n} \int_{\bar{C} \setminus C} W\underline{z} G^+(\underline{z} - \underline{z}_0)\, g(\underline{z}) - \frac{1}{\omega_n} \int_C D_M G^+(\underline{z} - \underline{z}_0)\, g(\underline{z})\, d\underline{z}^n. \quad \blacksquare$$

For the special case where g is the restriction to M of a holomorphic function, then the previous result was deduced in [11].

A simple application of Stokes' theorem gives us:

THEOREM 14. *Suppose M is a simple manifold of type one, $g\colon M \to A_n(\mathbb{C})$ is a continuous function which is C^1 on $\mathring{M}$, and $D_M g = 0$. Then for each point $\underline{z}_0 \in \mathring{M}$ and each manifold N of type one lying in M with $\underline{z}_0 \in \mathring{N}$, we have that*

$$g(\underline{z}_0) = \frac{1}{\omega_n} \int_{\partial N} G^+(\underline{z} - \underline{z}_0)\, W\underline{z} g(\underline{z}).$$

COROLLARY 3. *If g is as in Theorem 14, then g extends to a complex left-regular function on M^+.*

We now want to extend our results over simple manifolds of type one to more general manifolds of type one. First, let us consider the following simple example.

EXAMPLE 1. Let $M = \{\lambda(\cos\theta e_1 + i \sin\theta e_2), \lambda_2 e_2, ..., \lambda_n e_n\colon 0 \leqslant \lambda_j \leqslant 1$ for $1 \leqslant j \leqslant n$, and $-\pi/4 < \theta < \pi/4\}$. Then it may be observed that M is a manifold of type one, but it is not a simple manifold of type one. We would like to construct an operator

$$D_M\colon C^1(M, A_n(\mathbb{C})) \to C^0(M, A_n(\mathbb{C}))$$

such that if $g \in C^1(M, A_n(\mathbb{C}))$ is the restriction to M of some holomorphic function $\tilde{g}$ defined in a neighborhood of $\mathring{M}$, then

$$D_M g(\underline{z}) = D_{\mathbb{C}}\, \tilde{g}(\underline{z})\,|_{\mathring{M}}. \tag{23}$$

If we place $u = \cos\theta e_1 + i\sin\theta e_2$, and we allow $\partial_u g$ to denote the partial derivative of g in the direction of u, then if g has a C^1-extension g' to a neighborhood of $\mathring{M}$, we have that $\partial_u g(\underline{z}) = (\cos\theta(\partial/\partial x_1) + \sin\theta(\partial/\partial y_2)\, g'(\underline{z})$ for each $\underline{z} \in \mathring{M}$. Now as $\tilde{g}$ is holomorphic, we have that

$$\frac{\partial}{\partial y_2}\tilde{g}(\underline{z}) = i\frac{\partial}{\partial x_2}\tilde{g}(\underline{z})$$

for each $\underline{z} \in \mathring{M}$. So, $(\partial/\partial x_1)\,\tilde{g}(\underline{z} = (1/\cos\theta)(\partial_u - i\sin\theta(\partial/\partial x_2)\,\tilde{g}(\underline{z})$. Consequently, we may place

$$D_M = \frac{e_1}{\cos\theta}\left(\partial_u - i\sin\theta\frac{\partial}{\partial x_2}\right) + \sum_{j=2}^{n} e_j\frac{\partial}{\partial x_j},$$

for this particular choice of M. It is now straightforward to deduce that $D_M g(\underline{z}) = D_{\mathbb{C}}\tilde{g}(\underline{z})$ whenever g has a holomorphic extension $\tilde{g}$.

From this simple example and the definition of a manifold of type one, it is now straightforward to use the tangent bundle TM of a manifold M of type one to introduce a differential operator

$$D_M\colon C^1(M, A_n(\mathbb{C})) \to C^0(M, A_n(\mathbb{C})) \tag{24}$$

which satisfies (23) for each $g \in C^1(M, A_n(\mathbb{C}))$ which extends to a holomorphic function in a neighborhood of $\mathring{M}$. Moreover, it is straightforward to check that the operator (24) coincides with the intrinsic Dirac operator that we have introduced over simple manifolds of type one. For this reason, the operator D_M is called an intrinsic Dirac operator over M. Using the fact that D_M satisfies expression (23), and simple properties of the differential form $W\underline{z}$ over a manifold of type one, then the proof of Theorem 13 is readily adapted to obtain:

THEOREM 15. *Suppose that M is a manifold of type one, and $g\colon \mathring{M} \to A_n(\mathbb{C})$ is a bounded C^1-function with a bounded derivative. Then for each $\underline{z}_0 \in \mathring{M}$ we have*

$$D_M T_M g(\underline{z}_0) = -g(\underline{z}_0).$$

By similar arugments to those used to establish Theorem 14 and its corollary we also have:

THEOREM 16. *Suppose M is a manifold of type one, $g\colon M \to A_n(\mathbb{C})$ is a continuous function which is C^1 on $\mathring{M}$, and $D_M g = 0$. Then g extends to a complex left-regular function on M^+.*

Suppose now that $a\colon \mathring{M} \to \mathbb{C}$ is a bounded C^1-function. Then we can consider the inhomogeneous intrinsic Dirac operator

$$D_{M,a}\colon C^1(M, A_n(\mathbb{C})) \to C^0(M, A_n(\mathbb{C}))\colon g \mapsto D_M g + (D_M a)\, g. \qquad (25)$$

Using the results so far obtained in this section, it may be observed that the results obtained earlier in this paper for the operator $D + b(x)$ carry through for the operator (25). From now on we shall restrict our attention to the operator D_M, even though our remaining results do straightforwardly extend to the inhomogeneous intrinsic Dirac operator (25).

Application for the Operator D_M

We begin this section by taking a closer look at $L^2(M, A_n(\mathbb{C}))$. We first establish:

Proposition 5. *Suppose that $f, g \in L^2(M, A_n(\mathbb{C}))$. Then*

$$\left| \int_M Tr(f(\underline{z})\, g^*(\underline{z}))\, d\underline{z}^n \right| < +\infty,$$

where TrZ denotes the identity component of $Z \in A_n(\mathbb{C})$, and $(z_0 + \cdots + z_{1,\ldots,n} e_1 \cdots e_n)^* = z_0 + \cdots + z_{1,\ldots,n} (-1)^n e_n \cdots e_1$.

Proof. First, we may observe that

$$\left| \int_M Tr(f(\underline{z})\, g^*(\underline{z}))\, d\underline{z}^n \right| \leqslant \left\| \int_M f(\underline{z})\, g^*(\underline{z})\, d\underline{z}^n \right\|.$$

As M is compact, it may be observed that there is a constant $C \in R^+$ such that

$$\left\| \int_M f(\underline{z})\, g^*(\underline{z})\, d\underline{z}^n \right\| \leqslant C \int_M \|f(\underline{z})\|\ \|g(\underline{z})\|\, dM.$$

The result now follows from the Cauchy–Schwarz inequality. ∎

From Proposition 5 we have that there is a well-defined quadratic form

$$Q\colon L^2(M, A_n(\mathbb{C})) \times L^2(M, A_n(\mathbb{C})) \to \mathbb{C}\colon f, g \mapsto \int_M Tr(f(\underline{z})\, g^*(\underline{z}))\, d\underline{z}^n. \qquad (26)$$

This quadratic form is not an inner product for $L^2(M, A_n(\mathbb{C}))$. Indeed, $Q(f, f)$ need not be real in general. The Hilbert space structure for $L^2(M, A_n(\mathbb{C}))$ comes from the inner product

$$\langle\ ,\ \rangle: L^2(M, A_n(\mathbb{C})) \times L^2(M, A_n(\mathbb{C})) \to \mathbb{C}: \langle f, g\rangle$$
$$: \int_M Tr(f(\underline{z})\, g^*(\underline{z}))\, dM,$$

where $\bar{Z}^* = \bar{z}_0 + \cdots \bar{z}_{1, \ldots, n}(-1)^n e_n \cdots e_1$, for $Z = z_0 + \cdots + z_{1, \ldots, n} e_1 \cdots e_n$. The boundedness of the integral appearing in expression (26) follows from similar arguments to those used to establish Proposition 5. We now deduce:

PROPOSITION 6. *Suppose that* $g \in L^2(M, A_n(\mathbb{C})) \cap C^1(\mathring{M}, A_n(\mathbb{C}))$, *g has a continuous extension to* ∂M, *and*

$$\int_M f^*(\underline{z})\, g(\underline{z})\, d\underline{z}^n = 0$$

for all $f \in B_r^2(M, A_n(\mathbb{C}))$. *Then there is a* C^1*-function* $h: \mathring{M} \to A_n(\mathbb{C})$ *such that*

(i) $D_M h = g$

and

(ii) $\lim_{\underline{z} \to \underline{z}_0} h(\underline{z}) = 0$ *for all* $\underline{z} \in \mathring{M}$ *and* $\underline{z}_0 \in \partial M$.

Proof. As g is a C^1-function, we have from Theorem 15 that

$$g(\underline{z}') = -D_M T_M g(\underline{z}')$$

for each $\underline{z}' \in \mathring{M}$. Consequently,

$$\int_M f^*(\underline{z})\, g(\underline{z})\, d\underline{z}^n = -\int_M f^*(\underline{z})\, D_M T_M g(\underline{z})\, d\underline{z}^n. \tag{27}$$

As g is a continuous function on M, we have that for each $\underline{z}_0 \in \partial M$ and each $\underline{z}' \in \mathring{M}$

$$\begin{aligned}
&\|T_M g(\underline{z}_0) - T_M g(\underline{z}')\| \\
&\quad \leqslant \|T_{B(\underline{z}_0, r)} g(\underline{z}_0)\| + \|T_{B(\underline{z}', r)} g(\underline{z}')\| \\
&\qquad + \|T_{B(\underline{z}_0, r)} g(\underline{z}')\| + \|T_{B(\underline{z}', r)} g(\underline{z}_0)\| \\
&\qquad + \|T_{M \backslash (B(\underline{z}_0, r) \cup B(\underline{z}', r))} g(\underline{z}_0) - T_{M \backslash (B(\underline{z}_0, r) \cup B(\underline{z}', r))} g(\underline{z}')\|.
\end{aligned}$$

Using (27) and elementary estimates for $\|T_{B(\underline{z}_0, r)}g(\underline{z}_0)\|$, $\|T_{B(\underline{z}', r)}g(\underline{z}')\|$, $\|T_{B(\underline{z}_0, r)}g(\underline{z}')\|$ and $\|T_{B(\underline{z}', r)}g(\underline{z}_0)\|$, it follows that $T_M g(\underline{z}_0)$ has a continuous extension to ∂M. We also denote the extended function by $T_M g$. Moreover, it may be easily deduced that $T_M g$ is Lipschitz continuous on M. Consequently, we may apply Stokes' theorem to the right-hand side of expression (27) to obtain

$$\int_M f^*(\underline{z})\, g(\underline{z})\, d\underline{z}^n = -\int_{\partial M} f^*(\underline{z})\; W\underline{z} T_M g(\underline{z}),$$

whenever $f(\underline{z})$ has a continuous extension to ∂M, and $f^*(\underline{z})\, D_M = 0$.

We now consider the special case where $f(z) = (1/\omega_n)\, G^+(\underline{z} - \underline{z}_1)$, and $N(\underline{z}_1) \cap M = \phi$. In this case, $f^*(\underline{z}) = -(1/\omega_n)\, G^+(\underline{z} - \underline{z}_1)$, and

$$\frac{1}{\omega_n}\int_{\partial M} G^+(\underline{z} - \underline{z}_1)\; W\underline{z} T_M g(\underline{z})$$

is well-defined. From the Plemelj formulae in $\mathbb{C}^n$ given in [13] and Theorem 11, it now follows that as $T_M g$ is Lipschitz continuous, then

$$P.V. \frac{1}{\omega_n}\int_{\partial M} G^+(\underline{z} - \underline{z}')\; W\underline{z} T_M g(\underline{z}) = \frac{1}{2}\, T_M g(\underline{z})$$

for all smooth points $\underline{z}'$ on ∂M. Again by the Plemelj formulae given in Theorem 11 and [13], we have that

$$\lim_{\underline{z}' \to \underline{z}_1} \frac{1}{\omega_n}\int_{\partial M} G^+(\underline{z} - \underline{z}')\; W\underline{z} T_M g(\underline{z}) = T_M g(\underline{z}_1^*)$$

for $\underline{z}' \in \mathring{M}$ and for almost all $\underline{z}_1 \in \partial M$. On placing $h(\underline{z}') = T_M g(\underline{z}') - (1/\omega_n) \int_{\partial M} G^+(\underline{z} - \underline{z}')\; W\underline{z} T_M g(\underline{z})$, the result follows. ∎

For the special case where $M \subseteq R^n$, and M has a Liapunov boundary, this result appears in [6, Ch. 3]. We now deduce:

PROPOSITION 7. *Suppose that $M \subseteq \mathbb{C}^n$ is a manifold of type one, and h: $M \to A_n(\mathbb{C})$ is such that*

(i) *$h(\underline{z}) = 0$ for all $\underline{z} \in \partial M$.*

(ii) *h extends to a complex harmonic function h^+ on M^+, or its double covering* (so, $\sum_{j=1}^n (\partial^2 h^+/\partial z_j^2)(\underline{z}) = 0$).

(iii) *$D_{\mathbb{C}} h^+(\underline{z})$ has a continuous extension from $\mathring{M}$ to ∂M. Then h^+ is identically zero.*

Proof. By the complex extension of Green's formula [12], we have that for each $\underline{z}_0 \in \mathring{M}$

$$h(\underline{z}_0) = \frac{1}{\omega_n} \int_{\partial M} G^+(\underline{z} - \underline{z}_0)\, W\underline{z} h(\underline{z}) - \frac{1}{\omega_n} \int_{\partial M} H(\underline{z} - \underline{z}_0)\, W\underline{z} D_{\mathbb{C}} h^+(\underline{z}), \tag{28}$$

where $D_{\mathbb{C}} H(\underline{z}) = G^+(\underline{z})$. As $h(\underline{z}) = 0$ on ∂M, then expression (28) reduces to

$$h(\underline{z}_0) = -\frac{1}{\omega_n} \int_{\partial M} H(\underline{z} - \underline{z}_0)\, W\underline{z} D_{\mathbb{C}} h^+(\underline{z}). \tag{29}$$

Suppose now that M' is an n-dimensional manifold of type one lying in $\mathring{M}$, and $\underline{z}_0 \in \mathring{M}'$. Then as $h^+(\underline{z})$ is a complex harmonic function, we have from Stokes' theorem that

$$-\frac{1}{\omega_n} \int_{\partial M} H(\underline{z} - \underline{z}_0)\, W\underline{z} D_{\mathbb{C}} h^+(\underline{z}) + \frac{1}{\omega_n} \int_{\partial M'} H(\underline{z} - \underline{z}_0)\, W\underline{z} D_{\mathbb{C}} h^+(\underline{z})$$
$$= -\frac{1}{\omega_n} \int_{M \backslash M'} G^+(\underline{z} - \underline{z}_0)\, D_{\mathbb{C}} h^+(\underline{z})\, d\underline{z}^n. \tag{30}$$

On substituting (29) in (30) we obtain

$$h(\underline{z}_0) + \frac{1}{\omega_n} \int_{\partial M'} H(\underline{z} - \underline{z}_0)\, W\underline{z} D_{\mathbb{C}} h^+(\underline{z})$$
$$= -\frac{1}{\omega_n} \int_{M \backslash M'} (G^+(\underline{z} - \underline{z}_0)\, D_{\mathbb{C}} h^+(\underline{z})\, d\underline{z}^n.$$

Consequently,

$$D_{\mathbb{C}} h(\underline{z}_0) + \frac{1}{\omega_n} D_{\mathbb{C}} \int_{\partial M} H(\underline{z} - \underline{z}_0)\, W\underline{z} D_{\mathbb{C}} h^+(\underline{z}) = 0.$$

But

$$\frac{1}{\omega_n} D_{\mathbb{C}} \int_{\partial M} H(\underline{z} - \underline{z}_0)\, W\underline{z} D_{\mathbb{C}} h^+(\underline{z}) = D_{\mathbb{C}} h^+(\underline{z}_0),$$

as $D_{\mathbb{C}} h^+(\underline{z})$ is a complex left-regular function. Consequently,

$$2D_{\mathbb{C}} h(\underline{z}_0) = 0.$$

As this is true for each $\underline{z}_0 \in \mathring{M}$, we have that the extension of $D_{\mathbb{C}} h^+(\underline{z})$ to ∂M is identically zero. It now follows from (29) that $h(\underline{z}_0) = 0$ for all $\underline{z}_0 \in \mathring{M}$. Consequently, h^+ is identically zero on M^+. ∎

When M is a subset of R^n, then the previous result can also be proved using the maximum principle.

We have the following two corollaries to proposition 7:

Corollary 4. *Suppose M, $\subseteq \mathbb{C}^n$, is a manifold of type one, and $h_1(\underline{z})$ and $h_2(\underline{z})$ are complex harmonic functions on M^+ which both have continuous extensions to the same function on ∂M. Moreover, $D_C(h_1 - h_2)(\underline{z})$ extends continuously to ∂M. Then $h_1(\underline{z}) = h_2(\underline{z})$ for all $\underline{z} \in M^+$.*

Corollary 5. *Suppose M, $\subseteq \mathbb{C}^n$, is a manifold of type one, $h_1(\underline{z})$ and $h_2(\underline{z})$ are complex harmonic functions on M^+, and*

(i) $h_1(\underline{z}) = h_2(\underline{z})$ *for all* $\underline{z} \in \mathring{M}$

(ii) $D_{\mathbb{C}}(h_1(\underline{z}) - h_2(\underline{z}))$ *extends continuously to* ∂M.

Then $h_1(\underline{z}) = h_2(\underline{z})$ for all $\underline{z} \in M^+$.

Note. Using Corollary 5, the Riesz representation theorem may now be applied to introduce an analogue of harmonic measure over ∂M for each manifold M of type one lying in $\mathbb{C}^n$. This approach is a direct mimic of the way in which harmonic measure is introduced over Dirichlet domains in R^n. However, the lack, so far, of a maximum principle over ∂M for general M means that in order to obtain a bounded linear functional, one also needs to restrict attention to the $A_n(\mathbb{C})$-module $\mathscr{H}(\mathring{M}, A_n(\mathbb{C}))$ of $A_n(\mathbb{C})$-valued harmonic functions on M^+ which not only have continuous extensions to ∂M, but also $D_{\mathbb{C}} h(\underline{z})$ has a continuous extension to ∂M, for each $h \in \mathscr{H}_f(\mathring{M}, A_n(\mathbb{C}))$. In this case, Green's formula may be used to show the boundedness of the desired functionals acting over this module. This restriction seems to be somewhat artificial, and would easily be removed by a proper maximal principle over each manifold M of type one.

Let $C^{1,*}(M, A_n(\mathbb{C}))$ denote the $A_n(\mathbb{C})$-module of C^1-functions on $\mathring{M}$ which have continuous extensions to ∂M. Also, let $W(M, A_n(\mathbb{C}))$ be the $A_n(\mathbb{C})$-module of functions

$$g: M \to A_n(\mathbb{C})$$

satisfying the following properties:

(i) $g \in C^{1,*}(\mathring{M}, A_n(\mathbb{C}))$,

(ii) for each $g \in W(M, A_n(\mathbb{C}))$ there is a function $h_g: M \to A_n(\mathbb{C})$ such that $h_g|_{\partial M} = 0$ and $h_g \in C^1(\mathring{M}, A_n(\mathbb{C}))$.

Then from Propositions 6 and 7 we obtain:

THEOREM 17. *Suppose that M is a manifold of type one. Then*

$$L^2(M, A_n(\mathbb{C})) \cap C^{1,*}(M, A_n(\mathbb{C})) = (B_r^2(M, A_n(\mathbb{C})) \cap C^{1,*}(M, A_n(\mathbb{C}))) \oplus W(M, A_n(\mathbb{C})).$$

Moreover,

$$\int_{\partial M} Tr(f(\underline{z})\, g^*(\underline{z}))\, d\underline{z}^n = 0$$

for each $f \in B_r^2(M, A_n(\mathbb{C}))$ and each $g \in W(M, A_n(\mathbb{C}))$.

For the case where M is a subset of R^n, and $A_n(\mathbb{C})$ is replaced by A_n, then an analogue of this theorem appears in [6, Ch. 3]. However, Theorem 17 differs from its Euclidean analogue because the Hilbert space inner product over $B_r^2(M, A_n(\mathbb{C}))$ is replaced by the quadratic form (26) over general manifolds of type one. In the Euclidean setting, these two quadratic forms coincide. It is also worth noting that the condition that the functions appearing in Propositions 6 and 7, and Theorem 17 extend continuously from $\mathring{M}$ to ∂M is not really necessary. Those results carry through if each function on $\mathring{M}$ extends to an essentially bounded function on ∂M.

Theorem 17 enables us to introduce similar projection operators on $L^2(M, A_n(\mathbb{C}))$ to those used in [6, Ch. 3]. We may now obtain the following analogue of Dirichlet's problem:

THEOREM 18. *Suppose M is a manifold of type one, and $g(\underline{z})$ is an essentially bounded function on ∂M and is such that g has an extension to a C^1-function on $\mathring{M}$. Then there is a unique complex harmonic function $h: M^+ \to A_n(\mathbb{C})$ (or defined on the double cover of M^+) such that*

$$\lim_{\underline{z} \to \underline{z}'} h(\underline{z}) = g(\underline{z}')$$

for $\underline{z} \in \mathring{M}$ and almost all $\underline{z}' \in \partial M$.

We also have:

THEOREM 19. *Suppose M is a manifold of type one, $g \in C^1(\mathring{M}, A_n(\mathbb{C}))$ and g has an essentially bounded extension to almost all ∂M. Then there is a C^2-function $q: \mathring{M} \to A_n(\mathbb{C})$ such that*

(i) $-D_M^2 q(\underline{z}) = g(\underline{z})$ *for all* $\underline{z} \in \mathring{M}$

and

(ii) $\lim_{\underline{z} \to \underline{z}'} q(\underline{z}) = 0$ *for all* $\underline{z} \in \mathring{M}$ *and all* $\underline{z}' \in \partial M$.

From Theorems 18 and 19 we have:

THEOREM 20. *Suppose M is a manifold of type one, and $g(\underline{z})$, $q(\underline{z})$ are essentially bounded functions on ∂M which have extensions to C^1-functions on $\mathring{M}$. Then there is a function $h\colon \mathring{M} \to A_n(\mathbb{C})$ such that*

(i) $-D_M^2 h(\underline{z}) = g(\underline{z})$ *for all* $\underline{z} \in \mathring{M}$

and

(ii) $\lim_{\underline{z} \to \underline{z}'} h(\underline{z}) = q(\underline{z}')$ *for all* $\underline{z} \in \mathring{M}$ *and all* $\underline{z}' \in \partial M$.

This last result extends a classic result in Euclidean space [6, Ch. 4], to the special types of totally real manifolds considered here. Using the results developed earlier in this paper, we may see that Theorems 18, 19, and 20 all have straightforward extensions for the inhomogeneous intrinsic Dirac operators described in the previous section.

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